



Theta Modeling Technology with Mathematical Functions and Numerical Techniques  
respective an Underlying State Variable, Theta, wherein, an investment or derivative security is modeled by using a single underlying state variable, such a theta variable, insured deposits closed. These have a positive nominal value, an actual amount of cash value. This underlying state variable, theta,  $\theta$ , is held to follow an independent Markov process:  $d\theta/\theta = m dt + s dz$ .

This asserts that the future value of  $\theta$  depends on the known present values under a continuous pricing constraint. As Wiener process,  $dz$  is related to  $dt$ :  $\Delta z = \epsilon \sqrt{\Delta t}$ .

Such theta variable depends solely on itself and time to define its expected drift and volatility, which it redefines through the course of its life. Thus,  $d\theta/\theta = m(\theta, t) dt + s(\theta, t) dz$ .

For methods drawing from standard normal distribution, the log of the change of theta over time and/or the log of theta at exercise should have this distribution. The theta approach is useful where a target variable is not the price of a traded security, but it is useful there, too.

To creating a tradable instrument for a theta variable, assigning the function,  $f$ , as the price of a security dependent on  $\theta$  and time. For instance, for the insured banking's variable, the "deposits closed" and "deposit loss" are candidates for industry's theta. For the insured catastrophe dollar risk, they are the "catastrophe loss" and "net statutory underwriting loss". Variables are created from divers theta, such  $\theta_i$ , e.g.  $\theta_b$  and  $\theta_c$ , correlating respective losses.

For instance, for  $\theta$ (banking: of insured deposits closed or net deposit losses) and  $\theta$  (cat: of insured catastrophe or net statutory underwriting losses), let  $f(b)$  and  $f(c)$  be the respective price of a derivative security with payoff equal to a functional mapping of  $\theta_b$  and  $\theta_c$  into the future. Let the processes of  $f(b)$  and  $f(c)$  be defined via Ito's lemma, where

$$df/f = \mu dt + \sigma dz. \text{ This stands for any } f(\theta).$$

On a continuous time basis, the change in the price of security (f) dependent on the banking losses is  $df(b) = \mu_b f_b dt + \sigma_b f_b dz$ ; (and for  $\theta$ :  $df(c) = \mu_c f_c dt + \sigma_c f_c dz$ ). An instantaneously riskless portfolio can be created from a combination of related  $f(b_i)$ , such that  $(\mu_1 - r)/\sigma_1 = (\mu_2 - r)/\sigma_2 = \lambda$ , where  $r$  = the present spot, risk-free interest rate at time 1 and 2.

Thus, for any  $f$ , being the price of a security dependent on only  $\theta$  and time, with

$df = \mu f dt + \sigma f dz$ , there is the parameter  $\lambda$ ,  $\lambda = (\mu - r)/\sigma$ , which is dependent on  $\theta$  and time, but not on the security  $f$ , estimating the market pricing of risk of  $\theta$  by stochastics.

The theta variable's  $\mu$  is the expected return from  $f$ . The expected drift,  $\mu$ , equals  $\mu f$ . Sigma,  $\sigma f$ , is the volatility of  $f(\theta)$ , and either positively ( $df/d\theta > 0$ ) or negatively correlates to  $\theta$ . If negatively correlated, volatility =  $-\sigma$ , and  $df = \mu dt + (-\sigma) f (-dz)$ . Variance is  $[(\sigma^2)(f^2)]$  and  $dz$  is over an independent interval,  $dt = (T-t)$ . Using Ito's lemma, the parameter,  $\mu$ , relating  $\mu$  and the pricing function, is set as  $\mu^* f = df/dt + m \theta df/d\theta + \frac{1}{2} s^2 \theta^2 d^2f/d\theta^2$ .

The parameter Sigma is set,  $\sigma^* f = s \theta df/d\theta$ . This results in a differential structure

$$df/dt + \theta df/d\theta (m - \lambda s) + \frac{1}{2} s^2 \theta^2 d^2f/d\theta^2 = r^* f.$$

This equation can be solved by setting the drift of  $\theta$  equal to  $(m - \lambda s)$ , and discounting expected payoffs at  $r$ , the present spot risk-free (usu. U.S. Treasury) interest rate.

Thus, under risk-neutral valuation, the drift of  $\theta$  is reduced from  $m$ , to  $(m - \lambda s)$ .

To constructing a valuation lattice, such in discrete time, introducing the notions of

$$\delta, \delta = e^{(\sigma^* \sqrt{\Delta t})}, \text{ and of } \mu = [2^* e^{(r^* \Delta t)}] / [\delta + \delta^{-1}].$$

Hence,  $\sigma = \ln(\delta)/(\sqrt{\Delta t})$ . Next, setting values for  $\Delta t$ ,  $\sigma$ ,  $r(t)$  and  $\theta$  (where,  $\theta = S$ , if modeling an equity security), calculating nodes of  $\theta$  at  $\theta(tk) = [\mu^k] * [\delta^{w(k(w))}] * [\theta_0]$ .

Substituting theta ( $\theta$ ) for a security (S), S isomorphic to  $\theta$ , affording an underlying random walk of  $w(k(w))$ , such that if  $w=(-1,-1,1\dots)$ ,  $\theta(t(3)) = [\mu^3] * [\delta^{(-1)}] * [\theta_0]$ .

In lognormal world, this relates:  $\ln \theta(tn) = [n * \ln \mu] + [w(n(w)) * \ln \delta] + [\ln \theta_0]$ .  
This results in the equalities:  $\ln \delta = \sigma * (\sqrt{\Delta t})$  and  $dt = T/n$ . Substituting and letting  $k = n$ , such that  $tn = T$ , forms:  $\ln \theta(tk) = [n * \ln \mu] + [w(n(w)) * (\sigma * \sqrt{\Delta t}) / \sqrt{n}] + [\ln \theta_0]$ .

More simply, the expected value of  $\theta$ ,  $E(\ln \theta) = \ln \mu + \ln \theta_0$ .

The variance of  $\theta$ ,  $\text{Var}(\ln \theta) = (\ln \delta)^2$ .

The volatility of  $\theta$ ,  $\text{Vol}(\ln \theta) = (\ln \delta) / \sqrt{\Delta t}$ .

For a pathing tree, the node value mechanic,  $\theta(tn) = [\mu^n] * [\delta^{w(n(w))}] * [\theta_0]$ , using logarithmic transform, node mechanic,  $\ln \theta(tn) = [n * \ln \mu] + [w(n(w)) * (\ln \delta)] + [\ln \theta_0]$ .

By the Central Limit Theorem, the term,  $w(n(w)) / (\sqrt{n})$  exhibits strong convergence to the standard normal distribution,  $N(0,1)$ . The term  $[n * \ln \mu]$  shows weak convergence to  $[(r - \frac{1}{2} \sigma^2) * T]$ , hence its robust implementation is limited to the rigors of discrete methods.

The term  $[\ln \theta(T)]$  is distributed as  $[(r - \frac{1}{2} \sigma^2) * T] + [N * \sigma * (\sqrt{dt})] + [\ln \theta_0]$ .

Non-log,  $[\theta(T)]$  is distributed as:  $[\theta_0 * e^{((r - \frac{1}{2} \sigma^2) * T)} + (N * \sigma * (\sqrt{\Delta t}))]$ .

Valuation of a derivative security (S) based upon the state variable theta, example, the European call option, with realizable cashflow only at T, value today of P, with the functional mapping,  $f(\theta_0) = \max [\theta(T) - K, 0]$ , where K is strike price and  $\theta$  is held substitutable by S. By weak convergence, today's value,  $P(\theta)$ , based on  $\theta$  at T, derived over normal distribution:

$$P(\theta) = [\theta_0 * N\{(rT + \ln(\theta_0/K)) / (\sigma * (\sqrt{dt})) + (1/2 \sigma * (\sqrt{dt}))\}] -$$

$$[K e^{(-rT)} * N\{(rT + \ln(\theta_0/K)) / (\sigma * (\sqrt{dt})) - (1/2 \sigma * (\sqrt{dt}))\}].$$

$P = e^{(-rT)} * E(m)[\theta(T) - K]$ , where  $E(m)$  = expected value under risk-neutral conditions.

For any function,  $f$ , valuing a derivative security based on  $\theta$  that pays off  $f(T)$  at time  $T$ , the expected risk-neutral value is  $f = e^{(-rT)} * E(rn)(f(T))$ . This requires setting the growth rate of the underlying  $\theta$  variable in relation to  $[m - \lambda * \sigma]$ , rather than as  $m$  alone.

Thus, risk-neutral valuation for today's value,  $P(\theta)$ ,  $P(\theta)=f(\theta)$ , of a derivative security paying off  $f(T)$  at time  $T$ , is equivalent to the risk-free discount over period  $(0,T)$  of its expected risk-neutral future pay-out. This narrow evaluation is valid for  $f$  only over the continuous segment  $(0,T)$ , with determinable values of  $F(0)$  and  $F(T)$ . Lattices which subdivide this segment, are weakened, if their  $\Delta t$ -parameters, i.e.  $\Delta t = (T-t)$  with  $0 < t < T$ , are modeled using analytic values from  $(0,T)$  data sets. Any methodology which relies on convergence to a normal distribution for its valuation, for instance, or a sampling therefrom, is strictly consistent only for European-style derivatives, that is, having exercise only at  $T$ , but not continuously throughout the segment  $(0,T)$ . Also, it assumes the security can gain or lose value during  $(0,T)$ , with the value of the security always non-negative. The payoffs of the  $\theta_b$  and  $\theta_c$  securities can be European, if these stem from the single terminal condition of  $\theta$  at  $T$ : the selected  $\theta$  variables are annual aggregates, they begin each year at  $\theta=0$  and end the year at  $T$ ,  $\theta \geq 0$ , European  $(0,T)$  events. For rigorous risk-neutral valuation, strict conformity can only be assigned under a European-style  $(0,T)$  segment, variable and  $\theta$ -based security. Such a  $\theta$  can substitute for a continuously traded security after adjusting for the conditions that  $\theta$  at  $T = \sum \theta_i$ , each  $\theta_i$  occurring and aggregating discretely over  $(0,T)$ . For continuous trading, full data along the annual path of such  $\theta$  over  $(0,T)$  are required to be available.

A valuation function,  $V$ , for a security or derivative dependent only on  $\theta$  and time:

$$V(t,0)=V(t)= V_0 * e^{\{(r - \frac{1}{2} \sigma^2)t + (N * \sigma * (\sqrt{t}))\}}, \text{ where } V(0,T) \text{ is identity of } \theta(0,T).$$

To constructing a swap, e.g. between the insured deposit losses and catastrophe losses, modeled on  $\theta(b)$  and  $\theta(c)$  respectively, each having valuation function,  $f(b)$  and  $f(c)$  respectively. The value of the swap to the payer of the deposit losses,  $f(b)$ , assuming the swap of all year-end aggregate losses:

$$V = e^{(-rT)} * \{E(rn)[f(c)(T) - f(b)(T)]\}, \text{ where } E(rn) = \text{risk-neutral Expectation.}$$

Though a swap is composed of two sides, its value,  $V$ , is a single function. Thus,  $V$  is a single derivative instrument modeled by the expectation of the two functions, each respective of its own single  $\theta$  variable. Consequently, this function,  $V$ , models a security dependent on unrelated underlying variables,  $\theta(i)$ . Each  $\theta(i)$  follows a stochastic process of form:  $d\theta/\theta = \mu_i dt + \sigma_i dz_i$ , with  $\mu_i$  and  $\sigma_i$  the expected growth and volatility rates, the  $dz_i$  being Weiner processes, then substituting  $V$  for  $f$ , the total loss swap,  $V$ , has the form:

$$dV/V = \mu dt + \sum[\sigma_i dz_i], \text{ with } \mu \text{ being the expected return of the swap.}$$

Component risk of the return to the  $\theta(i)$ ,  $\sum[\sigma_i dz_i]$  are adjusted if  $\theta(i)$  are correlated.

Brownian motion defines the change in the value of a variable as related to the variable's initial value and characteristic deviation, as well as to distinct random perturbations resonating variance over independent intervals. It is a discrete process that approaches continuous form when the intervals are small and uncorrelated. Pinned simulation fixes an initial and terminal value for the variable, then developing the value path in between.

An expression of  $\theta$  with respect to time and to Brownian motion, the life of  $\theta$  over  $(0, T)$  and projected Brownian motion in simulation, can be related in derivational form:

$$\theta(t, B) = \theta_0 * e^{[(r - \frac{1}{2} \sigma^2)t + \sigma B]}.$$

This above equation values without preference to risk, obtaining risk-neutral results.

For geometric Brownian motion, the theta variable must be lognormal in functionality (i.e. its natural log values must show distribution in line with a standard normal population). To implementing this when modeling theta of such distribution, as respective of time only:

$$\theta(t) = \theta_0 * e^{[(r - \frac{1}{2} \sigma^2)t + \sigma \sqrt{t}]}.$$

The weak convergence by  $\Delta\theta(t)$ , requires only that the natural log of the change in theta shows a normalized distribution and characteristic variance (not necessarily  $\sigma^2=1$ ).

Allowing the notation,  $\theta(t,B) = \theta_0 * e^{[(r - \frac{1}{2} \sigma^2)t + \sigma B]}$ , the change in  $d\theta$ , measured at the terminal values  $(0,T)$ , with  $t=T-0$ , can be derived as:  $d\theta = \theta(B)dB + [\theta(t) + \frac{1}{2} \theta(BB)]dt$ ;

its partial derivative input parameters:

$$\theta(B) = d/dB \text{ of } \theta(t,B) = \sigma \theta;$$

$$\theta(BB) = \sigma^2 * \theta; \text{ and } \theta(t) = [r - \frac{1}{2} \sigma^2] * \theta.$$

This computes as  $d\theta = \sigma \theta dB + [(r - \frac{1}{2} \sigma^2)\theta + (\frac{1}{2} \sigma^2)\theta]dt$ , and can be reduced to:

$$d\theta = \sigma \theta dB + r \theta dt.$$

For normal theta variables, respective only to time and theta:

$$\theta(t) = \theta_0 * e^{[(r - \frac{1}{2} \sigma^2)t + (N \sigma \sqrt{t})]}; \text{ and}$$

$$V(0,T) \text{ as the identity of } \theta(0,T), V(t,0)=V(t)= V_0 * e^{[(r - \frac{1}{2} \sigma^2)t + (N \sigma \sqrt{t})]}.$$

This requires only that the natural log of the change in theta, hence, in  $V$ , has a characteristic, normal distribution.  $N$  represents a sampling off the standard normal distribution, e.g.  $N=\epsilon=\phi$ .

Monte Carlo simulation is a discrete methodology that is based on the Law of Large Numbers (e.g. in large numbers of sampling sequences). The life of the security is subdivided into  $n$  intervals, each of length  $\Delta t$ . Using  $s \equiv$  volatility, and  $m \equiv$  risk-neutral growth rate, of  $\theta$ :

$$\Delta\theta = m \theta \Delta t + s \theta \epsilon(\sqrt{\Delta t}), \text{ where each simulation run has } n \text{ drawings, one per } \Delta t.$$

For a multiple state  $\theta$ :  $\Delta\theta_i = m_i\theta_i\Delta t + s_i\theta_i \epsilon_i(\sqrt{\Delta t})$ , with  $\theta_i$ : ( $1 \leq i \leq n$ ). If the  $\theta_i$  are correlated, implement correlation between the  $\theta_i$ ,  $\rho_{ik}$ , and also between the  $\epsilon_i$  and  $\rho_{ik}$ .

The Figure 91 diagrams the mathematical programming functions for theta variable. The Figure 92 diagrams the mathematical programming functions for security based on theta.

For further embodiment of theta, and as respective reinsurance and actuarial sciences, see inventor's published work, "Forecasting Expectations of Insured Depository Default and Catastrophic Losses". The publication further elaborates the purposes and mechanics of theta.

Regarding the use of a put-option model for deposit guarantees, i.e. insurance against deposit losses, a deposit insurance guarantee is not isomorphic to an European put option, as any recovery of underwritten (collateralizing) assets, occurs later than the deposit loss occurs. Asset recovery or loss development does not occur simultaneously to default or catastrophe.

An OAS/martingale lattice, featuring the development of recovery or loss over time, implemented in diagram schematic, Figure 93, an Option Adjusted Spread style lattice. Having an initial node, at  $t = 0$ , at which node having value at given  $t$ , expressed as  $e(0) = 1$ . From initial node, binary pathing, to the upper node at time  $t = 1$ , processing  $1 - M(0)h(1)$ , wherein  $M$  is martingale for the given  $h$ , and wherein  $h$  is distinct time interval, value at upper node of  $e(1) = 1$ , a state wherein no default or loss occurs; to the lower node at time  $t = 1$ , processing  $M(0)h(1)$ , value at lower node of  $e(1) = E(L)$ , a state where default or loss occurs, wherein  $E(L)$  is the risk-neutral expectation of  $L$  after asset recovery or loss development.

From the upper node at time  $t = 1$ , binary pathing: to an upper node at time  $t = 2$ , processing  $1 - M(1)h(2)$ , value at upper node of  $e(2) = 1$ , wherein no default or loss occurs; to a lower node at time  $t = 2$ , processing  $M(1)h(2)$ , value at lower node of  $e(2) = E(L)$ , a state where default or loss occurs. If lower node occurs, follow path relation of lower node at  $t = 1$ .